

# SOME NON-RELATIVISTIC IMPLICATIONS OF HYDROGENIC SOLUTIONS TO ANGULAR MOMENTA

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The intention of this paper is to provide solutions to commutative relations relevant to calculations regarding the hydrogen atom (or similar mono-electronic systems). Though exact solutions exist to these systems, the value to approximation methods stems from the ability to conveniently parse physical non-sense by comparison with these identities. A derivation accompanies each identity.

## 1 Identities

Since

$$\vec{L} = \vec{r} \times \vec{p}$$

Then

$$\begin{vmatrix} L_x & L_y & L_z \\ p_x & p_y & p_z \\ x & y & z \end{vmatrix}$$

Thus:

$$L_x = yp_z - zp_y$$

$$L_y = -xp_z + zp_x$$

$$L_z = xp_y - yp_x$$

### 1.1 $[L_x, x]$

$$[L_x, x] = L_x x - x L_x = (yp_z - zp_y)x - x(yp_z - zp_y)$$

Apply a test function:

$$[L_x, x]\Psi = (yp_z - zp_y)x\Psi - x(yp_z - zp_y)\Psi \Rightarrow (-i\hbar) \left[ y \frac{\partial}{\partial z}(x\Psi) - z \frac{\partial}{\partial y}(x\Psi) \right] - (-i\hbar) \left[ xy \frac{\partial}{\partial z}(\Psi) - xz \frac{\partial}{\partial y}(\Psi) \right]$$

No partial derivatives are affecting  $x$ , thus:

$$(-i\hbar) \left[ xy \frac{\partial}{\partial z}(\Psi) - zx \frac{\partial}{\partial y}(\Psi) \right] + (i\hbar) \left[ xy \frac{\partial}{\partial z}(\Psi) - xz \frac{\partial}{\partial y}(\Psi) \right]$$

$x, y, z$  all commute, and, in general,  $[p_j, i], [i, p_j] = 0$  for  $i \neq j$ . Thus:

$$(-i\hbar) \left[ xy \frac{\partial}{\partial z}(\Psi) - zx \frac{\partial}{\partial y}(\Psi) \right] + (i\hbar) \left[ xy \frac{\partial}{\partial z}(\Psi) - xz \frac{\partial}{\partial y}(\Psi) \right] = (yx p_z - zx p_y) - (xy p_z - zx p_y) = 0$$

Thus,

$$[L_x, x] = 0$$

## 1.2 $[L_x, y]$

$$[L_x, y] = L_x y - y L_x = (y p_z - z p_y) y - y (y p_z - z p_y)$$

Apply a test function:

$$\begin{aligned} [L_x, y]\Psi &= (y p_z - z p_y) y \Psi - y (y p_z - z p_y) \Psi \Rightarrow (-i\hbar) \left[ y \frac{\partial}{\partial z} (y \Psi) - z \frac{\partial}{\partial y} (y \Psi) \right] - (-i\hbar) \left[ y^2 \frac{\partial}{\partial z} (\Psi) - y z \frac{\partial}{\partial y} (\Psi) \right] \\ &= (-i\hbar) \left[ y^2 \frac{\partial}{\partial z} (\Psi) - z \left( y \frac{\partial \Psi}{\partial y} + \Psi \right) \right] - (-i\hbar) \left[ y^2 \frac{\partial}{\partial z} (\Psi) - y z \frac{\partial}{\partial y} (\Psi) \right] = (-i\hbar) \left[ y^2 \frac{\partial \Psi}{\partial z} - z y \frac{\partial \Psi}{\partial y} - z \Psi - y^2 \frac{\partial \Psi}{\partial z} + y z \frac{\partial \Psi}{\partial y} \right] \\ &= (-i\hbar) (-z \Psi) = [L_x, y] \Psi \end{aligned}$$

Thus,

$$[L_x, y] = z i \hbar$$

## 1.3 $[L_x, z]$

$$[L_x, z] = L_x z - z L_x = (y p_z - z p_y) z - z (y p_z - z p_y)$$

Apply a test function:

$$\begin{aligned} [L_x, z]\Psi &= (y p_z - z p_y) z \Psi - z (y p_z - z p_y) \Psi \Rightarrow (-i\hbar) \left[ y \frac{\partial}{\partial z} (z \Psi) - z \frac{\partial}{\partial y} (z \Psi) \right] - (-i\hbar) \left[ z y \frac{\partial}{\partial z} (\Psi) - z^2 \frac{\partial}{\partial y} (\Psi) \right] \\ &= (-i\hbar) \left[ y \left( z \frac{\partial \Psi}{\partial z} + \Psi \right) - z^2 \left( \frac{\partial \Psi}{\partial y} \right) \right] - (-i\hbar) \left[ z y \frac{\partial}{\partial z} (\Psi) - z^2 \frac{\partial}{\partial y} (\Psi) \right] = (-i\hbar) \left[ z y \frac{\partial \Psi}{\partial z} - z^2 \frac{\partial \Psi}{\partial y} + y \Psi - z^2 \frac{\partial \Psi}{\partial y} - y z \frac{\partial \Psi}{\partial z} \right] \\ &= (-i\hbar) (y \Psi) = [L_x, y] \Psi \end{aligned}$$

Thus,

$$[L_x, z] = -y i \hbar$$

## 1.4 $[L_x, p_x]$

$$\begin{aligned} &L_x p_x \Psi - p_x L_x \Psi \\ &= (y p_z p_x - z p_y p_x) \Psi - (p_x y p_z - p_x z p_y) \Psi \end{aligned}$$

Term-by-term:

$$\begin{aligned} y p_z p_x \Psi &= (-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial z \partial x} \\ -z p_y p_x \Psi &= -(-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y \partial x} \\ p_x y p_z \Psi &= (-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial y \partial x} \\ -p_x z p_y \Psi &= -(-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial x \partial y} \end{aligned}$$

By Clairaut's Theorem, the order of differentiation is immaterial; thus:

$$(-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial z \partial x} - (-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y \partial x} - (-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial y \partial x} + (-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial x \partial y} = 0$$

### 1.5 $[L_x, p_y]$

$$L_x p_y \Psi - p_y L_x \Psi \Rightarrow (y p_z p_y - z p_y p_y) \Psi - (p_y y p_z - p_y z p_y) \Psi$$

$$y p_z p_y \Psi = (-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial z \partial y}$$

$$-z p_y p_y \Psi = -(-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y^2}$$

$$p_y y p_z \Psi = (-i\hbar) \left[ p_z + y \frac{\partial p_z}{\partial y} \right] \Psi$$

$$-p_y z p_y \Psi = -(-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y^2}$$

Or:

$$\begin{aligned} & (-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial z \partial y} - (-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y^2} - (-i\hbar) \left[ p_z + y \frac{\partial p_z}{\partial y} \right] \Psi + (-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y^2} \\ & -(-i\hbar) p_z \Psi = -(-i\hbar)^2 \frac{\partial \Psi}{\partial z} \Rightarrow [L_x, p_y] = (i\hbar) p_z \end{aligned}$$

### 1.6 $[L_x, p_z]$

$$L_x p_z \Psi - p_z L_x \Psi \Rightarrow (y p_z p_z - z p_y p_z) \Psi - (p_z y p_z - p_z z p_y) \Psi$$

$$y p_z p_z \Psi = (-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial z^2}$$

$$-z p_y p_z \Psi = -(-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y \partial z}$$

$$p_z y p_z \Psi = (-i\hbar)^2 y \frac{\partial^2 \Psi}{\partial z^2}$$

$$-p_z z p_y \Psi = (-i\hbar) \left[ p_y + z \frac{\partial p_y}{\partial z} \right] \Psi$$

The first and third terms cancel (after distributing the negation, and we are left with:

$$-(-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y \partial z} - (-i\hbar) p_y \Psi + (-i\hbar)^2 z \frac{\partial^2 \Psi}{\partial y \partial z}$$

Thus:

$$[L_x, p_z] \Psi = -(-i\hbar) p_y \Psi \Rightarrow [L_x, p_z] = (i\hbar) p_y$$

### 1.7 $[L_x, L_y]$

$$(y p_z - z p_y)(-x p_z + z p_x) - (-x p_z + z p_x)(y p_z - z p_y)$$

$$y p_z z p_x - y p_z x p_z - z p_y z p_x + z p_y x p_z - (z p_x y p_z - z p_x z p_y - x p_z y p_z + x p_z z p_y) =$$

$$y p_z z p_x - y p_z x p_z - z p_y z p_x + z p_y x p_z - z p_x y p_z + z p_x z p_y + x p_z y p_z - x p_z z p_y$$

We now know which terms commute, so collect the ones that do not:

$$[y p_z z p_x - x p_z z p_y] \Psi$$

$$y(-i\hbar) \left[ z \frac{\partial p_x}{\partial z} + p_x \right] \Psi - x(-i\hbar) \left[ z \frac{\partial p_y}{\partial z} + p_y \right] \Psi$$

$$yz(-i\hbar)^2 \frac{\partial^2 \Psi}{\partial x \partial z} + y(-i\hbar) p_x \Psi - xz(-i\hbar)^2 \frac{\partial^2 \Psi}{\partial z \partial y} - y(-i\hbar) p_y \Psi$$

Note that

$$[A - B, C - D] = [A, C] - [D, A] - [B, C] + [B, D]$$

So

$$[yp_z - zp_y, zp_x - xp_z]$$

Is

$$[yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$$

And by similar argument, the middle terms will eventually vanish towards the end of the calculation, so we neglect them now:

$$[yp_z, zp_x] + [zp_y, xp_z] = yp_x[z, p_z] + xp_y[z, p_z] = i\hbar(xp_y - yp_x) = i\hbar L_z$$

Since

$$yp_x zp_z - yp_x p_z z + xp_y zp_z - xp_y p_z z$$

### 1.8 $[L_x, \vec{L}^2]$

$$[L_x, L_x^2] + [L_x, L_y^2] + [L_x, L_z^2]$$

$$[L_x, L_x^2] = 0$$

So

$$[L_x, L_y]L_y + L_y[L_x, L_y]L_y + [L_x, L_z]L_z + L_z[L_x, L_z]$$

$$(-i\hbar L_z)L_y + L_y(-i\hbar L_z) + (-i\hbar L_y)L_z + L_z(-i\hbar L_y)$$

Since

$$[AB, C] = A[B, C] + [A, C]B = ABC - ACB + ACB - CAB = ABC - CAB = [AB, C]$$

Then:

$$(-i\hbar L_z)L_y + L_y(-i\hbar L_z) + (-i\hbar L_y)L_z + L_z(-i\hbar L_y) = 0$$

### 1.9 $[L_x, \vec{r}^2]$

$$[L_x, x^2] + [L_x, y^2] + [L_x, z^2]$$

$$[L_x, x^2] = L_x x^2 - x^2 L_x = 0$$

So

$$L_x y^2 - y^2 L_x + L_x z^2 - z^2 L_x$$

### 1.10 $[L_x, \vec{p}^2]$

$$[L_x, p_x^2] + [L_x, p_y^2] + [L_x, p_z^2]$$

$$[L_x, L_x^2] = 0$$

So

## 2 Theorem: Skew Hermitian operators have pure imaginary eigenvalues

*Proof.* If

$$\langle \hat{\Omega} \Psi | \Psi \rangle = -\langle \Psi | \hat{\Omega} \Psi \rangle$$

And

$$\langle \hat{\Omega} \Psi | \Psi \rangle = \Omega^* \langle \Psi | \Psi \rangle$$

$$\langle \Psi | \hat{\Omega} \Psi \rangle = \langle \Psi | \Psi \rangle \Omega$$

Then, by definition, the skew-hermitian operator  $\hat{\Omega}$  satisfies:

$$\Omega^* \langle \Psi | \Psi \rangle = -(\langle \Psi | \Psi \rangle \Omega) \Rightarrow \Omega^* = -\Omega$$

If  $\Omega \in \mathbb{C}$ , then it assumes the form  $A + iB$  where  $A, B \in \mathbb{R}$  and

$$\Omega^* = -\Omega \Rightarrow (A + iB)^* = -(A + iB)$$

Or

$$A - iB = -A - iB$$

Which can only be true if  $A = 0$ . Thus, the observables (eigenvalues) from skew Hermitian operators are always purely imaginary.  $\square$

## 3 References

1. Boas, Mary *Mathematical Methods for the Physical Sciences* Wiley, Sons, Co. 2002
2. Griffiths, D.J. *Introduction to Quantum Mechanics* Harper-Collins 2003